3051. [2005 : 333, 335] *Proposed by Vedula N. Murty, Dover, PA, USA.*

Let x, y, $z \in [0,1)$ such that $x+y+z=1$. Prove that

(a) $\sqrt{\frac{x}{x+yz}}+\sqrt{\frac{y}{y+zx}}+\sqrt{\frac{z}{z+xy}} \;\le\; 3\sqrt{\frac{3}{2}}$;

(b) $\frac{\sqrt{xyz}}{(1-x)(1-y)(1-z)} \;\le\; \frac{3\sqrt{3}}{8}$.

Similar solutions by Arkady Alt, San Jose, CA, USA; and Peter Y. Woo, Biola University, La Mirada, CA, USA, modified by the editor.

For part (a) we will prove the stronger inequality

$$\sqrt{\frac{x}{x+yz}}+\sqrt{\frac{y}{y+xz}}+\sqrt{\frac{z}{z+xy}} \;\le\; \frac{3\sqrt{3}}{2}\,. \tag{1}$$

We may assume for both parts (a) and (b) that x, y, $z \in (0,1)$, because the inequalities are trivial if one of x, y, z is zero. Our proof will be based on the following inequality, which is an immediate consequence of the convexity of the sine function on $[0,\pi]$:

$$\sin\theta_1+\sin\theta_2+\sin\theta_3 \;\le\; \frac{3\sqrt{3}}{2} \tag{2}$$

for all θ_1, θ_2, $\theta_3 > 0$ such that $\theta_1+\theta_2+\theta_3=\pi$.

Let $a=y+z$, $b=z+x$, and $c=x+y$. Since $x+y+z=1$, we have $a=1-x$, $b=1-y$, $c=1-z$, and $a+b+c=2$. The numbers a, b, c satisfy the triangle inequalities $a+b>c$, $b+c>a$, and $c+a>b$. Therefore, a, b, c are the lengths of the sides of a triangle. Let α, β, γ be the angles opposite the sides a, b, c, respectively. Thus, $\alpha+\beta+\gamma=\pi$. Note that the semiperimeter of the triangle is $s=\frac{1}{2}(a+b+c)=1$.

(a) We have

$$\begin{aligned}\frac{x}{x+yz} &= \frac{1-a}{1-a+(1-b)(1-c)} = \frac{1-a}{2-(a+b+c)+bc}\\ &= \frac{1-a}{bc} = \frac{s(s-a)}{bc} = \cos^2(\alpha/2)\,.\end{aligned}$$

Similarly, $\dfrac{y}{y+zx} = \cos^2(\beta/2)$ and $\dfrac{z}{z+xy} = \cos^2(\gamma/2)$. Thus, the left side of (1) is equal to $\cos(\alpha/2) + \cos(\beta/2) + \cos(\gamma/2)$. To prove (1), it will be sufficient to prove that

$$\cos(\alpha/2) + \cos(\beta/2) + \cos(\gamma/2) \;\le\; \frac{3\sqrt{3}}{2}$$

for all α, β, $\gamma > 0$ such that $\alpha + \beta + \gamma = \pi$. But this follows by applying (2) with $\theta_1 = \frac{1}{2}(\pi - \alpha)$, $\theta_2 = \frac{1}{2}(\pi - \beta)$, and $\theta_3 = \frac{1}{2}(\pi - \gamma)$.

(b) Let K and R be the area and circumradius, respectively, of the triangle with sides a, b, c. Then

$$\begin{aligned}
\frac{\sqrt{xyz}}{(1-x)(1-y)(1-z)} &= \frac{\sqrt{(s-a)(s-b)(s-c)}}{abc} = \frac{sK}{4KR} = \frac{s}{4R} \\
&= \frac{1}{4}\left(\frac{a}{2R} + \frac{b}{2R} + \frac{c}{2R}\right) \\
&= \frac{1}{4}(\sin\alpha + \sin\beta + \sin\gamma) \;\le\; \frac{3\sqrt{3}}{8}\,,
\end{aligned}$$

where the last step follows by applying (2) with $\theta_1 = \alpha$, $\theta_2 = \beta$, and $\theta_3 = \gamma$.

Also solved by MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; RONGZHENG JIAO, Yangzhou University, Yangzhou, China; PHAM VAN THUAN, Hanoi University of Science, Hanoi, Vietnam; PANOS E. TSAOUSSOGLOU, Athens, Greece; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and the proposer.

All solvers proved the stronger inequality for (a) *which appears in the featured solution.*